

# ELEMENTARY SYMMETRIC POLYNOMIALS IN STANLEY-REISNER FACE RING

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**ABSTRACT.** Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets. In this paper we pay our attention on those elementary symmetric polynomials in the Stanley–Reisner face ring of  $P$  and study how the decomposability of the  $n$ -th elementary symmetric polynomial influences on the combinatorics of  $P$  and the topology and geometry of toric spaces over  $P$ . We give algebraic criterions of detecting the decomposability of  $P$  and determining when  $P$  is  $n$ -colorable in terms of the  $n$ -th elementary symmetric polynomial. In addition, we define the Stanley–Reisner *exterior* face ring  $\mathcal{E}(K_P)$  of  $P$ , which is non-commutative in the case of  $\mathbb{Z}$  coefficients, where  $K_P$  is the boundary complex of dual of  $P$ . Then we obtain a criterion for the (real) Buchstaber invariant of  $P$  to be  $m - n$  in terms of the  $n$ -th elementary symmetric polynomial in  $\mathcal{E}(K_P)$ . Our results as above can directly associate with the topology and geometry of toric spaces over  $P$ . In particular, we show that the decomposability of the  $n$ -th elementary symmetric polynomial in  $\mathcal{E}(K_P)$  with  $\mathbb{Z}$  coefficients can detect the existence of the almost complex structures of quasitoric manifolds over  $P$ , and if the (real) Buchstaber invariant of  $P$  is  $m - n$ , then there exists an essential relation between the  $n$ -th equivariant characteristic class of the (real) moment-angle manifold over  $P$  in  $\mathcal{E}(K_P)$  and the characteristic functions of  $P$ .

## 1. INTRODUCTION

The Stanley–Reisner face ring is a fundamental tool in algebraic combinatorics and combinatorial commutative algebra ([15], [19]). In addition, it is well-known that the Stanley–Reisner face ring can also be realizable as the equivariant cohomology of many toric spaces over a simple polytope  $P$ , such as toric varieties, (quesi-)toric manifolds, small covers and (real) moment-angle manifolds. Moreover, elementary symmetric polynomials in the Stanley–Reisner face ring can be realized as the equivariant Chern classes or equivariant Stiefel–Whitney classes of those toric spaces. Thus, the Stanley–Reisner face ring also plays an essential role on algebraic geometry, toric geometry and toric topology ([4]).

Let  $K$  be an abstract simplicial complex of dimension  $n - 1$  on vertex set  $[m] = \{1, \dots, m\}$ , and let  $R$  be a commutative ring with unit. Then the *Stanley–Reisner face ring*  $R(K)$  of  $K$  is defined as the quotient ring of the polynomial ring  $R[x_1, \dots, x_m]$

$$R(K) = R[x_1, \dots, x_m] / \mathcal{I}_K$$

where  $\mathcal{I}_K$  is the ideal generated by those square-free monomials  $x_{i_1} \cdots x_{i_r}$  for which  $\{i_1, \dots, i_r\} \notin K$ . Let

$$\sigma_i^K(x_1, \dots, x_m) = \sum x_{j_1} \cdots x_{j_i}$$

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be the  $i$ -th elementary symmetric polynomial function in Stanley–Reisner face ring  $R(K)$ , which is the image of the standard  $i$ -th elementary symmetric polynomial function  $\sigma_i(x_1, \dots, x_m)$  in  $R[x_1, \dots, x_m]$  under the natural projection

$$p : R[x_1, \dots, x_m] \longrightarrow R(K) = R[x_1, \dots, x_m]/\mathcal{I}_K.$$

In general,  $\sigma_i^K(x_1, \dots, x_m)$  is a deficient function in  $R(K)$  since some monomials of  $\sigma_i(x_1, \dots, x_m)$  in  $R[x_1, \dots, x_m]$  may be missing in  $R(K)$ . However, these  $\sigma_i^K(x_1, \dots, x_m)$  record the complete combinatorics of  $K$ . In fact, for two abstract simplicial complexes  $K_1, K_2$  of dimension  $n - 1$  on  $[m]$ , we can regard  $R(K_1)$  and  $R(K_2)$  as two quotient rings of  $R[x_1, \dots, x_m]$ . Then it is easy to see that  $K_1$  and  $K_2$  are combinatorially isomorphic if and only if there is a permutation  $s$  on  $\{x_1, \dots, x_m\}$  such that  $\sigma_i^{K_1}(x_1, \dots, x_m) = \sigma_i^{K_2}(s(x_1), \dots, s(x_m))$  for all  $i$ .

In this paper we shall investigate the internal implications (e.g., decomposability) of those polynomials  $\sigma_i^K(x_1, \dots, x_m)$  in some different rings, such as  $R(K)$ ,  $R[x_1, \dots, x_m]$  and the Stanley–Reisner exterior face ring etc. Our main purpose is to study how the internal implications of those polynomials  $\sigma_i^K(x_1, \dots, x_m)$  can produce essential influences on the combinatorics of  $K$  and the topology and geometry of toric spaces. Here we will pay more attention on the case in which  $K$  is the boundary complex  $K_P$  of the dual polytope of a simple  $n$ -polytope  $P$  with  $m$  facets although our many arguments can still be carried on for the case of more general  $K$ . This is because in this case we can directly associate with the topology and geometry of toric spaces over  $P$ , and in particular,  $\sigma_n^{K_P}(x_1, \dots, x_m)$  completely determines the combinatorics of  $K_P$ .

Given an  $n$ -dimensional simple polytope  $P$  with  $m$  facets, we shall carry out our work in the following some aspects:

- (1) We show that the decomposability of  $P$  agrees with that of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R[x_1, \dots, x_m]$  (see Theorem 3.7), where the decomposability of  $P$  means whether  $P$  is a product of some polytopes or not, and we note that there is a natural module embedding  $R(K_P) \hookrightarrow R[x_1, \dots, x_m]$ , so  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$  can be regarded as a polynomial in  $R[x_1, \dots, x_m]$ . This gives an algebraic criterion of detecting the decomposability of  $P$  in terms of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R[x_1, \dots, x_m]$ . It should be pointed out that the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$  is different from that of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R[x_1, \dots, x_m]$ . In other words, we cannot use the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$  to detect that of  $P$ . This work is motivated by [12, Section 7, Problem (P2)]. Here we give an answer to [12, Section 7, Problem (P2)] in spite of the existence of characteristic functions on  $P$ .
- (2) With a combinatorial argument, we give a criterion for  $P$  to be  $n$ -colorable in terms of the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$  (see Theorem 4.2). More generally, for each integer  $\ell \geq n$ , we can also give a criterion for  $P$  to be  $\ell$ -colorable in terms of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$  (see Theorem 4.4). With a topological argument, Notbohm in [16, 17] has given a criterion for  $P$  to be  $\ell$ -colorable in terms of the splitting property or total characteristic class of a vector bundle. However, our criterion is a little bit weaker than Notbohm's one. In fact, Notbohm's criterion needs to involve all elementary symmetric polynomials  $\sigma_i^{K_P}(x_1, \dots, x_m)$ ,  $i \leq n$ , but our criterion only needs the  $n$ -th elementary symmetric polynomial  $\sigma_n^{K_P}(x_1, \dots, x_m)$ .

- (3) We study the Buchstaber invariant  $s_{\mathbb{C}}(P)$  (or  $s_{\mathbb{R}}(P)$ ) of  $P$ , which is a combinatorial invariant, introduced by Buchstaber, and has also the geometrical meaning ([3]). Actually,  $s_{\mathbb{C}}(P)$  (or  $s_{\mathbb{R}}(P)$ ) is related to the existence of the free actions of the maximal subtori on moment-angle manifold  $\mathcal{Z}_P^{\mathbb{C}}$  (or real moment-angle manifold  $\mathcal{Z}_P^{\mathbb{R}}$ ) over  $P$ . In a similar way to the Stanley–Reisner face ring, we introduce the Stanley–Reisner *exterior* face ring  $\mathcal{E}_{\mathbb{F}}(K_P)$  over  $R_{\mathbb{F}}$  of  $P$ , where  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{R}$ , and  $R_{\mathbb{F}}$  is  $\mathbb{Z}$  if  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{Z}_2$  if  $\mathbb{F} = \mathbb{R}$ . This ring  $\mathcal{E}_{\mathbb{F}}(K_P)$  is non-commutative if  $\mathbb{F} = \mathbb{C}$ . Then we give a necessary and sufficient condition for  $s_{\mathbb{F}}(P) = m - n$  in terms of the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $\mathcal{E}_{\mathbb{F}}(K_P)$  (see Theorem 5.2 and Theorem 5.6), where if  $\mathbb{F} = \mathbb{C}$ ,  $\sigma_n^{K_P}(x_1, \dots, x_m)$  will be an oriented polynomial associated with an orientation of  $K_P$ .

Our work as above can directly associate with the topology and the geometry of toric spaces over  $P$  although most of our results and proofs are combinatorial and algebraic, so that we can obtain more understandings from the viewpoints of topology and geometry. In fact,  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R_{\mathbb{F}}(K_P)$  is exactly the  $n$ -th equivariant Chern class (or equivariant Stiefel–Whitney class) of  $\mathcal{Z}_P^{\mathbb{C}}$  (or  $\mathcal{Z}_P^{\mathbb{R}}$ ). Thus,  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R_{\mathbb{F}}[x_1, \dots, x_m]$  is the pullback of the  $n$ -th equivariant characteristic class  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R_{\mathbb{F}}(K_P)$  via the embedding  $R_{\mathbb{F}}(K) \hookrightarrow R_{\mathbb{F}}[x_1, \dots, x_m]$ . We will also see that if  $\mathbb{F} = \mathbb{C}$ , the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  can detect the existence of the almost complex structures of quasitoric manifolds over  $P$  (see Proposition 5.4), and in particular, we give a simple criterion for determining whether a quasitoric manifold admits an equivariant almost complex structure (see Corollary 5.5). In addition, our necessary and sufficient condition for  $s_{\mathbb{F}}(P) = m - n$  also implies that there exists an essential relation between the  $n$ -th equivariant characteristic class  $\sigma_n^{K_P}(x_1, \dots, x_m)$  of  $\mathcal{Z}_P^{\mathbb{C}}$  (or  $\mathcal{Z}_P^{\mathbb{R}}$ ) in  $\mathcal{E}_{\mathbb{F}}(K_P)$  and the characteristic functions of  $P$  (see Corollary 5.3 and Corollary 5.7). An application to the Buchstaber invariants of cyclic polytopes is also given.

## 2. GEOMETRIC REALIZATION OF THE STANLEY–REISNER FACE RING IN TORIC TOPOLOGY

**2.1. Moment-angle manifolds.** An  $n$ -dimensional polytope  $P$  is said to be *simple* if each vertex of  $P$  meets exactly  $n$  facets. Simple polytopes have played essential roles in the theory of toric varieties, toric geometry and toric topology. Let  $P$  be an  $n$ -dimensional simple polytope with  $m$  facets  $F_1, \dots, F_m$ . Then its dual  $P^*$  is a simplicial polytope of dimension  $n$ , and the boundary  $\partial P^*$  of  $P^*$ , denoted by  $K_P$ , is a simplicial complex of dimension  $n - 1$  on vertex set  $[m]$  (corresponding to the facet set  $\{F_1, \dots, F_m\}$  of  $P$ ). The complex  $K_P = \partial P^*$  is also called the *polytopal sphere* of  $P$ . It is well-known that over  $P$ , the moment-angle manifold  $\mathcal{Z}_P^{\mathbb{C}}$  and real moment-angle manifold  $\mathcal{Z}_P^{\mathbb{R}}$  can be defined with different ways. Following [3, Construction 6.38], for each simplex  $\sigma$  in  $K_P$ , set  $B_{\sigma} = \prod_{i=1}^m A_i$  such that

$$A_i = \begin{cases} D^2 & \text{if } i \in \sigma \\ S^1 & \text{if } i \in [m] \setminus \sigma \end{cases}$$

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where  $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$  is the unit disk in  $\mathbb{C}$ , and  $S^1 = \partial D^2$ . Then one can define the moment-angle manifold  $\mathcal{Z}_P^{\mathbb{C}}$  as the following subspace of the product space  $(D^2)^m$ :

$$\mathcal{Z}_P^{\mathbb{C}} := \bigcup_{\sigma \in K_P} B_{\sigma} \subset (D^2)^m.$$

The manifold  $\mathcal{Z}_P^{\mathbb{C}}$  admits a natural action of  $\mathbb{T}_{\mathbb{C}}^m = (S^1)^m$ , which is the restriction to  $\mathcal{Z}_P^{\mathbb{C}}$  of the standard  $\mathbb{T}_{\mathbb{C}}^m$ -representation on  $\mathbb{C}^m$  given by

$$((g_1, \dots, g_m), (z_1, \dots, z_m)) \mapsto (g_1 z_1, \dots, g_m z_m).$$

The complex conjugation on  $\mathbb{C}^m$  gives a conjugation involution on  $\mathcal{Z}_P^{\mathbb{C}}$  whose fixed point set is exactly the real moment-angle manifold  $\mathcal{Z}_P^{\mathbb{R}} \subset (D^1)^m \subset \mathbb{R}^m$  and admits an action of the elementary 2-group  $\mathbb{T}_{\mathbb{R}}^m = (S^0)^m \cong \mathbb{Z}_2^m$ , where  $\mathbb{T}_{\mathbb{R}}^m$  is the fixed point set of the complex conjugation on  $\mathbb{T}_{\mathbb{C}}^m \subset \mathbb{C}^m$ . As shown in [7],  $\mathcal{Z}_P^{\mathbb{R}}$  plays an important role on the study of quasitoric manifolds and small covers where  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{R}$ . We note that each  $\mathcal{Z}_P^{\mathbb{R}}$  itself can be defined in a similar way to  $\mathcal{Z}_P^{\mathbb{C}}$ , but it can always be realized as the fixed point set of the conjugation involution on  $\mathcal{Z}_P^{\mathbb{C}}$ . However, generally both  $\mathcal{Z}_P^{\mathbb{C}}$  and  $\mathcal{Z}_P^{\mathbb{R}}$  have quite differences, as shown in [7, Lemma 6.5]. One also knows from [7] that both  $\mathbb{T}_{\mathbb{C}}^m$ -action on  $\mathcal{Z}_P^{\mathbb{C}}$  and  $\mathbb{T}_{\mathbb{R}}^m$ -action on  $\mathcal{Z}_P^{\mathbb{R}}$  have the same orbit space  $P$ .

**2.2. Equivariant cohomology and equivariant characteristic classes of  $\mathcal{Z}_P^{\mathbb{F}}$ .** Davis and Januszkiewicz showed in [7] that the cohomology of the Borel construction  $E\mathbb{T}_{\mathbb{F}}^m \times_{\mathbb{T}_{\mathbb{F}}^m} \mathcal{Z}_P^{\mathbb{F}}$  of  $\mathcal{Z}_P^{\mathbb{F}}$  only depends upon  $P$ , and it is isomorphic to the Stanley–Reisner face ring of  $P$ .

**Theorem 2.1** ([7, Theorem 4.8]). *There is a ring isomorphism*

$$H_{\mathbb{T}_{\mathbb{F}}^m}^*(\mathcal{Z}_P^{\mathbb{F}}; R_{\mathbb{F}}) \cong R_{\mathbb{F}}(K_P) = R_{\mathbb{F}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$$

with  $\deg x_i = \dim \mathbb{F}$ , where  $R_{\mathbb{F}}$  is  $\mathbb{Z}$  if  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{Z}_2$  if  $\mathbb{F} = \mathbb{R}$ .

In [7], Davis and Januszkiewicz constructed  $m$  canonical line bundles  $\mathbb{L}_i (i = 1, \dots, m)$  over  $E\mathbb{T}_{\mathbb{F}}^m \times_{\mathbb{T}_{\mathbb{F}}^m} \mathcal{Z}_P^{\mathbb{F}}$ , which are stated as follows. First, let  $\rho_i : \mathbb{T}_{\mathbb{F}}^m \rightarrow \mathbb{T}_{\mathbb{F}}$  be the projection onto the  $i$ -th factor and let  $\mathbb{F}(\rho_i)$  denote the corresponding 1-dimensional  $\mathbb{T}_{\mathbb{F}}^m$ -representation space. Then, define a trivial equivariant line bundle  $\tilde{\mathbb{L}}_i$  over  $\mathcal{Z}_P^{\mathbb{F}}$  by  $\tilde{\mathbb{L}}_i = \mathbb{F}(\rho_i) \times \mathcal{Z}_P^{\mathbb{F}}$ . Finally, consider the Borel construction on  $\tilde{\mathbb{L}}_i$ , we may obtain the required line bundle

$$\mathbb{L}_i = E\mathbb{T}_{\mathbb{F}}^m \times_{\mathbb{T}_{\mathbb{F}}^m} \tilde{\mathbb{L}}_i$$

and the first Stiefel–Whitney class  $w_1(\mathbb{L}_i) = x_i$  if  $\mathbb{F} = \mathbb{R}$ , and first Chern class  $c_1(\mathbb{L}_i) = x_i$  if  $\mathbb{F} = \mathbb{C}$ . In fact,  $\mathbb{L}_i$  exactly corresponds to the facet  $F_i$  of  $P$ . Furthermore, Davis and Januszkiewicz showed in [7, Theorem 6.6] that the Borel construction  $E\mathbb{T}_{\mathbb{F}}^m \times_{\mathbb{T}_{\mathbb{F}}^m} \mathcal{T}\mathcal{Z}_P^{\mathbb{F}}$  of the tangent bundle  $\mathcal{T}\mathcal{Z}_P^{\mathbb{F}}$  is stably isomorphic to the Whitney sum  $\mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_m$  as real vector bundles. Thus, if  $\mathbb{F} = \mathbb{R}$ , the total equivariant Stiefel–Whitney class of  $\mathcal{T}\mathcal{Z}_P^{\mathbb{R}}$  is

$$w^{\mathbb{T}_{\mathbb{R}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{R}}) = \prod_{i=1}^m (1 + x_i) \text{ in } H_{\mathbb{T}_{\mathbb{R}}^m}^*(\mathcal{Z}_P^{\mathbb{R}}; \mathbb{Z}_2) \cong R_{\mathbb{R}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$$

(see also [7, Corollary 6.7]). If  $\mathbb{F} = \mathbb{C}$ , following the arguments of [4, Theorem 7.3.15], there is the following  $\mathbb{T}_{\mathbb{C}}^m$ -equivariant decomposition by restricting the tangent bundle  $\mathcal{T}\mathbb{C}^m$  to  $\mathcal{Z}_P^{\mathbb{C}}$

$$(2.1) \quad \mathcal{T}\mathcal{Z}_P^{\mathbb{C}} \oplus \nu(i_{\mathcal{Z}}) \cong \mathcal{Z}_P^{\mathbb{C}} \times \mathbb{C}^m$$

where  $i_Z : \mathcal{Z}_P^{\mathbb{C}} \hookrightarrow \mathbb{C}^m$  is a  $\mathbb{T}_{\mathbb{C}}^m$ -equivariant embedding, and  $\nu(i_Z)$  is the normal bundle of  $\mathcal{Z}_P^{\mathbb{C}}$  in  $\mathbb{C}^m$  which is  $\mathbb{T}_{\mathbb{C}}^m$ -equivariantly trivial. Since  $\mathcal{Z}_P^{\mathbb{C}} \times \mathbb{C}^m$  is isomorphic to  $\widetilde{\mathbb{L}}_1 \oplus \cdots \oplus \widetilde{\mathbb{L}}_m$  as  $\mathbb{T}_{\mathbb{C}}^m$ -equivariant vector bundles, applying the Borel construction to the bundle isomorphism (2.1), one has that

$$(E\mathbb{T}_{\mathbb{C}}^m \times_{\mathbb{T}_{\mathbb{C}}^m} \mathcal{T}\mathcal{Z}_P^{\mathbb{C}}) \oplus (E\mathbb{T}_{\mathbb{C}}^m \times_{\mathbb{T}_{\mathbb{C}}^m} \nu(i_Z)) \cong \mathcal{Z}_P^{\mathbb{C}} \times_{\mathbb{T}_{\mathbb{C}}^m} \mathbb{C}^m = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_m$$

so the total equivariant Chern class of  $\mathcal{T}\mathcal{Z}_P^{\mathbb{C}}$  is

$$c^{\mathbb{T}_{\mathbb{C}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{C}}) = \prod_{i=1}^m (1 + x_i) \text{ in } H_{\mathbb{T}_{\mathbb{C}}^m}^*(\mathcal{Z}_P^{\mathbb{C}}; \mathbb{Z}) \cong R_{\mathbb{C}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$$

and thus  $w_i^{\mathbb{T}_{\mathbb{R}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{R}})$  and  $c_i^{\mathbb{T}_{\mathbb{C}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{C}})$  can be written as  $\sigma_i^{K_P}(x_1, \dots, x_m)$ .

We note that since  $H^*(B\mathbb{T}_{\mathbb{F}}^m; R_{\mathbb{F}}) = R_{\mathbb{F}}[x_1, \dots, x_m]$ , the natural projection

$$p : R_{\mathbb{F}}[x_1, \dots, x_m] \longrightarrow R_{\mathbb{F}}(K_P) = R_{\mathbb{F}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$$

is actually induced by the fibration  $\pi : E\mathbb{T}_{\mathbb{F}}^m \times_{\mathbb{T}_{\mathbb{F}}^m} \mathcal{Z}_P^{\mathbb{F}} \longrightarrow B\mathbb{T}_{\mathbb{F}}^m$ , so  $\sigma_i^{K_P}(x_1, \dots, x_m)$  is the image of the  $i$ -th universal characteristic class  $\sigma_i(x_1, \dots, x_m)$  in  $R_{\mathbb{F}}[x_1, \dots, x_m]$  under the homomorphism  $\pi^*$ . Generally, each polynomial  $\bar{f}$  in  $R_{\mathbb{F}}(K_P)$  is a coset  $f + \mathcal{I}_{K_P}$  where  $f \in R_{\mathbb{F}}[x_1, \dots, x_m]$ . Since the ideal  $\mathcal{I}_{K_P}$  is exactly generated by square free monomials, each coset  $f + \mathcal{I}_{K_P}$  contains a unique representative that has no any monomial in  $\mathcal{I}_{K_P}$ . Such representative will be said to be *prime*. Then we have

**Lemma 2.2.** *There is a module embedding  $e : R_{\mathbb{F}}(K_P) \hookrightarrow R_{\mathbb{F}}[x_1, \dots, x_m]$  defined by mapping  $\bar{f}$  to its prime representative.*

Without any confusion, we will identify  $\bar{f}$  with its prime representative. With this understanding,  $R_{\mathbb{F}}(K_P)$  can be regarded as a submodule of  $R_{\mathbb{F}}[x_1, \dots, x_m]$ , and in particular,  $\sigma_i^{K_P}(x_1, \dots, x_m)$  in  $R_{\mathbb{F}}(K_P)$  can also understood as a polynomial in  $R_{\mathbb{F}}[x_1, \dots, x_m]$ .

**2.3. Buchstaber invariant.** In general, the action of  $\mathbb{T}_{\mathbb{F}}^m$  on  $\mathcal{Z}_P^{\mathbb{F}}$  is not free, but the action restricted to some sub-tori of  $\mathbb{T}_{\mathbb{F}}^m$  may be free. There is the maximum rank of those sub-tori that can freely on  $\mathcal{Z}_P^{\mathbb{F}}$ , denoted by  $s_{\mathbb{F}}(P)$ , which is called *Buchstaber invariant* in [3]. It was known from [3] that

**Proposition 2.3.**  $1 \leq s_{\mathbb{C}}(P) \leq s_{\mathbb{R}}(P) \leq m - n$ .

If  $\dim P = 3$ , by Four Color Theorem, then  $s_{\mathbb{C}}(P) = s_{\mathbb{R}}(P) = m - 3$ . Generally,  $s_{\mathbb{C}}(P)$  and  $s_{\mathbb{R}}(P)$  may be strictly less than  $m - n$ . This can be seen from cyclic polytopes of dimension  $\geq 4$ . Buchstaber invariant can also be defined for the general simplicial complexes, and it is a combinatorial invariant. However, the calculation of Buchstaber invariant is quite difficult and complicated ([3, Problem 7.27]). Some works on the properties of Buchstaber invariant and the calculations in some special cases have been carried on (see, e.g. [1, 2, 8, 9, 10, 11, 13]).

**2.4. Quasitoric manifolds and small covers.** In their seminar work [7], Davis and Januszkiewicz introduced and studied quasitoric manifolds and small covers. Let  $P$  be an  $n$ -dimensional simple polytope with  $m$  facets  $F_1, \dots, F_m$ . If Buchstaber invariant  $s_{\mathbb{F}}(P) = m - n$ , then we can choose a subtorus  $H_{\mathbb{F}}$  of rank  $m - n$  in  $\mathbb{T}_{\mathbb{F}}^m$  which acts freely on  $\mathcal{Z}_P^{\mathbb{F}}$ , so that the quotient space  $\mathcal{Z}_P^{\mathbb{F}} / H_{\mathbb{F}}$  is a closed manifold and admits an action of  $\mathbb{T}_{\mathbb{F}}^m / H_{\mathbb{F}} \cong \mathbb{T}_{\mathbb{F}}^n$ . The space  $\mathcal{Z}_P^{\mathbb{F}} / H_{\mathbb{F}}$  is called a *quasitoric manifold* if  $\mathbb{F} = \mathbb{C}$ , and a *small cover*

if  $\mathbb{F} = \mathbb{R}$ . Conversely, if there is a quasitoric manifold (or small cover) over  $P$ , then it is easy to see that  $s_{\mathbb{C}}(P) = m - n$  (or  $s_{\mathbb{R}}(P) = m - n$ ). Thus,

**Proposition 2.4.** *There exist quasirotic manifolds (resp. small covers) over  $P$  if and only if  $s_{\mathbb{C}}(P) = m - n$  (resp.  $s_{\mathbb{R}}(P) = m - n$ ).*

Let  $\pi_{\mathbb{F}} : M_{\mathbb{F}} \longrightarrow P$  be a quasitoric manifold or a small cover over  $P$ , where  $\dim M_{\mathbb{F}} = \dim \mathbb{F} \cdot n$  and  $M_{\mathbb{F}}$  admits an action of  $\mathbb{T}_{\mathbb{F}}^n$ . Davis and Januszkiewicz showed in [7] that the Stanley–Reisner face ring can also be realizable as the equivariant cohomology of  $M_{\mathbb{F}}$ .

**Theorem 2.5** ([7, Theorem 4.8]). *There is a ring isomorphism*

$$H_{\mathbb{T}_{\mathbb{F}}^n}^*(M_{\mathbb{F}}; R_{\mathbb{F}}) \cong R_{\mathbb{F}}(K_P) = R_{\mathbb{F}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$$

with  $\deg x_i = \dim \mathbb{F}$ , where  $R_{\mathbb{F}}$  is  $\mathbb{Z}$  if  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{Z}_2$  if  $\mathbb{F} = \mathbb{R}$ .

In addition, as shown in [7], the action of  $\mathbb{T}_{\mathbb{F}}^n$  on  $M_{\mathbb{F}}$  gives an essential information on  $P$  via  $\pi_{\mathbb{F}}$ , which is just the *characteristic function*

$$\lambda : \mathcal{F}(P) = \{F_1, \dots, F_m\} \longrightarrow R_{\mathbb{F}}^n$$

such that for each vertex  $v$  of  $P$  (so there are exactly  $n$  facets, say  $F_{i_1}, \dots, F_{i_n}$ , such that  $v = F_{i_1} \cap \dots \cap F_{i_n}$ ),  $\lambda(F_{i_1}), \dots, \lambda(F_{i_n})$  form a basis of  $R_{\mathbb{F}}^n$ . The characteristic function  $\lambda$  can naturally be spanned into a linear map  $R_{\mathbb{F}}^m \longrightarrow R_{\mathbb{F}}^n$ , also denoted by  $\lambda$ . Fixing an ordering of all facets of  $P$ , say  $F_1, \dots, F_m$ , then  $\lambda$  determines a unique  $(n \times m)$ -matrix  $\Lambda = (\lambda_{ij})$ . As shown in [7], the transpose of this matrix  $\Lambda = (\lambda_{ij})$  is actually identified with the map  $p^* : H^{\dim \mathbb{F}}(B\mathbb{T}_{\mathbb{F}}^n; R_{\mathbb{F}}) \longrightarrow H_{\mathbb{T}_{\mathbb{F}}^n}^{\dim \mathbb{F}}(M_{\mathbb{F}}; R_{\mathbb{F}})$ , where  $p^*$  is induced by the fibration  $p : E\mathbb{T}_{\mathbb{F}}^n \times_{\mathbb{T}_{\mathbb{F}}^n} M_{\mathbb{F}} \longrightarrow B\mathbb{T}_{\mathbb{F}}^n$ . Thus, the characteristic function  $\lambda$  can be regarded as a sequence

$$(\lambda_1, \dots, \lambda_n)$$

(still denoted by  $\lambda$ ) in  $H_{\mathbb{T}_{\mathbb{F}}^n}^*(M_{\mathbb{F}}; R_{\mathbb{F}}) \cong R_{\mathbb{F}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$ , where  $\lambda_i = \lambda_{i1}x_1 + \dots + \lambda_{im}x_m$ . The sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  determines an ideal  $J_{\lambda} = \langle \lambda_1, \dots, \lambda_n \rangle$  of  $R_{\mathbb{F}}(K_P)$ . Then

**Theorem 2.6** ([7, Theorem 4.14]). *The cohomology of  $M_{\mathbb{F}}$  is the quotient ring of  $R_{\mathbb{F}}(K_P)$  by  $J_{\lambda}$ . Namely*

$$H^*(M_{\mathbb{F}}; R_{\mathbb{F}}) \cong R_{\mathbb{F}}(K_P) / J_{\lambda}.$$

**Remark 1.** *As shown in [5], the stably complex structure on a quasitoric manifold  $M_{\mathbb{C}}$  depends upon the choice of omniorientations. For more details, see [5].*

### 3. DECOMPOSABILITY OF SIMPLE POLYTOPES

**3.1. Abstract simplicial complexes and their joins.** An abstract simplicial complex  $K$  on a finite set  $S$  is a collection in the power set  $2^S$  such that for each  $a \in K$ , any subset (including empty set) of  $a$  still belongs to  $K$ . Each  $a$  in  $K$  is called a simplex and has dimension  $|a| - 1$ , where  $|a|$  is the cardinality of  $a$ . The dimension of  $K$  is defined as  $\max_{a \in K} \{\dim a\}$ .

It is well-known (cf. [3]) that up to combinatorial equivalence, each finite abstract simplicial complex can be realized as a unique geometric simplicial complex; and conversely, each finite geometric simplicial complex gives a unique abstract simplicial complex. Thus, we may identify finite abstract simplicial complexes with finite

geometric simplicial complexes. Throughout the following, a simplicial complex will mean a finite abstract simplicial complex or a finite geometric simplicial complex.

Let  $K_1$  and  $K_2$  be two simplicial complexes on finite sets  $S_1$  and  $S_2$ , respectively. Then the *join* of  $K_1$  and  $K_2$  is an abstract simplicial complex

$$K_1 * K_2 = \{a \cup b \in 2^{S_1 \cup S_2} \mid a \in K_1, b \in K_2\}$$

on the set  $S_1 \cup S_2$ .

**3.2. Polynomials of simplicial complexes.** Let  $K$  be a simplicial complex on the set  $S$  with  $|S| = m$ . Without the loss of generality, assume that  $K$  is a simplicial complex on  $[m] = \{1, \dots, m\}$ . Now consider the commutative polynomial ring  $R[x_1, \dots, x_m]$  where  $R$  is a commutative ring with unit and the  $x_i$  are indeterminants of same degree. Regarded  $K$  as a poset with respect to the inclusion, for each maximal element  $a = \{i_1, \dots, i_r\}$  in  $K$ , we define a monomial  $x_{i_1} \cdots x_{i_r}$ , denoted by  $m_a$ . Then we obtain a square-free polynomial

$$f_K = \sum_a m_a$$

where  $a$  runs over all maximal elements in  $K$ . We call  $f_K$  the *polynomial of  $K$* .

If  $K$  is the boundary complex  $K_P$  of the dual polytope of an  $n$ -dimensional simple polytope  $P$  with  $m$  facets, then  $f_{K_P}$  is a square-free homogeneous polynomial in  $R[x_1, \dots, x_m]$ , and it is exactly the  $n$ -th elementary symmetric function  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R_{\mathbb{F}}(K_P) \hookrightarrow R_{\mathbb{F}}[x_1, \dots, x_m]$ .

A polynomial  $f$  in  $R[x_1, \dots, x_m]$  is said to be *nice* if the coefficients of all monomials of  $f$  are 1 and each monomial of  $f$  is not a factor of other monomials of  $f$ . Let  $f = \sum x_{i_1} \cdots x_{i_r}$  be a nice polynomial in  $R[x_1, \dots, x_m]$ . Clearly each monomial  $x_{i_1} \cdots x_{i_r}$  determines a subset  $\{i_1, \dots, i_r\}$  of  $[m]$ , called the *monomial subset* of  $x_{i_1} \cdots x_{i_r}$ . Then we obtain a simplicial complex  $K_f$  with all monomial subsets as its maximal elements. This gives

**Lemma 3.1.** *All simplicial complexes on  $[m]$  bijectively correspond to all nice polynomials in  $R[x_1, \dots, x_m]$ .*

**Proposition 3.2.** *Let  $K$  be a simplicial complex on  $[m]$ . Then  $K$  is a join of two simplicial complexes if and only if  $f_K$  is a product of two polynomials in  $R[x_1, \dots, x_m]$ .*

*Proof.* If  $K = K_1 * K_2$ , then it is easy to see that  $f_K = f_{K_1} f_{K_2}$ . Conversely, if  $f = f_1 f_2$ , since  $f$  is square free, both  $f_1$  and  $f_2$  are square free, too. Moreover, both  $f_1$  and  $f_2$  are nice, and  $K_{f_1}, K_{f_2}$  share no common vertices. Then we see easily that  $K_f = K_{f_1} * K_{f_2}$ .  $\square$

**3.3. The decomposability of  $P$ .** Now let us discuss the decomposability of a simple polytope  $P$ .

**Lemma 3.3** ([3]). *Let  $P_1$  and  $P_2$  be simple polytopes. Then*

$$K_{P_1} * K_{P_2} = K_{P_1 \times P_2}.$$

**Lemma 3.4.** *If  $K$  is a polytopal sphere, then the link of any vertex of  $K$  is also a polytopal sphere.*

*Proof.* Assume  $K = \partial P^*$  where  $P$  is a simple polytope with the facet set  $\mathcal{F}(P) = \{F_1, F_2, \dots, F_m\}$ . We note that  $K$  may be regarded as the simplicial complex with  $\mathcal{F}(P)$  as its vertex set such that each simple  $\sigma$  of  $K$  is a subset  $\{F_{i_1}, \dots, F_{i_r}\}$  with  $F_{i_1} \cap \dots \cap F_{i_r} \neq \emptyset$  of  $\mathcal{F}(P)$ . With this understood, now given a vertex  $v$  in  $K$ , then there is a facet  $F$  of  $P$  such that  $v = \{F\}$ . Thus

$$\begin{aligned} \text{Link}_K(v) &= \{\tau = \{F_{i_1}, \dots, F_{i_r}\} \in K \mid v \cup \tau \in K, v \cap \tau = \emptyset\} \\ &= \{\{F_{i_1}, \dots, F_{i_r}\} \in K \mid F \cap F_{i_1} \cap \dots \cap F_{i_r} \neq \emptyset\}. \end{aligned}$$

Clearly,  $\text{Link}_K(v)$  is combinatorially equivalent to the simplicial complex  $K' = \{\{F \cap F_{i_1}, \dots, F \cap F_{i_r}\} \mid (F \cap F_{i_1}) \cap \dots \cap (F \cap F_{i_r}) \neq \emptyset, F \cap F_{i_r} \in \mathcal{F}(F)\}$  with  $\mathcal{F}(F)$  as vertex set, where  $\mathcal{F}(F)$  denotes the set of all facets of  $F$ . Since  $P$  is simple,  $F$  is also simple, so  $K'$  is the boundary complex  $\partial F^*$  of the dual of  $F$ . Therefore,  $\text{Link}_K(v)$  is a polytopal sphere.  $\square$

**Lemma 3.5.** *Let  $K'$  be a simplicial complex and  $K$  be the suspension of  $K'$  such that  $K = K' * S^0$ , where  $S^0$  is the 0-dimensional sphere. Then  $K$  is a polytopal sphere if and only if  $K'$  is a polytopal sphere.*

*Proof.* We note that  $S^0$  is the polytopal sphere of the interval  $I$  as a 1-dimensional polytope. So  $S^0 = \partial I^*$ . Let  $v_1$  and  $v_2$  be two vertices of  $I$ . Then  $S^0$  is the 0-dimensional simplicial complex on vertex set  $\{v_1, v_2\}$ . Furthermore,  $K = K' * S^0 = \{\tau, \tau \cup \{v_1\}, \tau \cup \{v_2\} \mid \tau \in K'\}$ , i.e., the suspension of  $K'$ .

If  $K$  is a polytopal sphere, obviously  $\text{Link}_K(v_1) = \text{Link}_K(v_2) = K'$ . By Lemma 3.4,  $K'$  is a polytopal sphere. Conversely, if  $K'$  is a polytopal sphere, we may assume that  $K' = \partial P'^*$  for some simple polytope  $P'$ . By Lemma 3.3, we have  $K = K' * S^0 = (\partial P'^*) * (\partial I^*) = \partial(P' \times I)^*$ . So  $K$  is exactly the polytopal sphere of  $P' \times I$ .  $\square$

**Lemma 3.6.** *Let  $K, K_1$  and  $K_2$  be simplicial complexes such that  $K = K_1 * K_2$ . If  $K$  is a polytopal sphere, then both  $K_1$  and  $K_2$  are polytopal spheres, too.*

*Proof.* We will perform an induction on the dimension of  $K$ . We note that  $\dim K = \dim K_1 + \dim K_2 + 1 \geq 1$ .

When  $\dim K = 1$ , since  $K$  is a polytopal sphere, it must be a circle, so it would be a boundary complex of the dual of a polygon. It is not hard to check that  $K_1 = K_2 = S^0$ . So the theorem follows.

Now assume inductively that the theorem holds if  $\dim K \leq n$ . When  $\dim K = n + 1$ , by Lemma 3.5 we know that  $K * S^0 = K_1 * K_2 * S^0$  is a polytopal sphere too. Take a vertex  $v$  of  $K_1$ ,  $v$  is also a vertex of  $K$ . Consider the link of  $v$  in  $K * S^0$ , we have

$$\text{Link}_{K * S^0}(v) = \text{Link}_{K_1 * K_2 * S^0}(v) = (\text{Link}_{K_1}(v)) * K_2 * S^0.$$

Using Lemmas 3.4 and 3.5, since  $K * S^0$  is polytopal sphere, both  $\text{Link}_{K * S^0}(v)$  and  $(\text{Link}_{K_1}(v)) * K_2$  are so. Since  $\dim((\text{Link}_{K_1}(v)) * K_2) = \dim K_1 * K_2 - 1 = n$ , by induction hypothesis,  $K_2$  is a polytopal sphere. In a similar way as above, choose a vertex of  $K_2$ , we can prove that  $K_1$  is also a polytopal sphere.  $\square$

With Proposition 3.2 and Lemma 3.6 together, we have

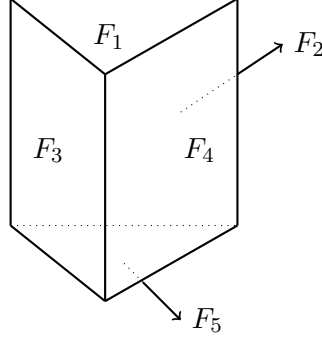
**Theorem 3.7.** *Let  $P$  be an  $n$ -dimensional simple polytope with  $m$  facets. Then  $P$  is indecomposable (i.e.,  $P$  is not a product of polytopes) if and only if the polynomial  $f_{K_P} = \sigma_n^{K_P}(x_1, \dots, x_m)$  is indecomposable in  $R[x_1, \dots, x_m]$ .*



Associated with the equivariant characteristic classes of the moment-angle  $\mathcal{Z}_P^{\mathbb{F}}$  over  $P$ , we have

**Corollary 3.8.** *An  $n$ -dimensional simple polytope  $P$  with  $m$  facets is indecomposable if and only if the  $n$ -th equivariant Stiefel–Whitney class  $w_n^{\mathbb{T}_{\mathbb{R}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{R}})$  or Chern class  $c_n^{\mathbb{T}_{\mathbb{C}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{C}})$  is indecomposable in  $H^*(B\mathbb{T}_{\mathbb{F}}^m; R_{\mathbb{F}}) = R_{\mathbb{F}}[x_1, \dots, x_m]$ .*

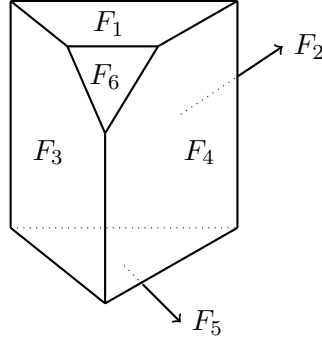
**Example 1.**  $P$  is a prism which is the product  $\Delta^1 \times \Delta^2$  of two simplices and has 5 facets, shown as follows.



Then we see that

$$\begin{aligned}\sigma_3^{K_P}(x_1, \dots, x_5) &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 \\ &= (x_1 + x_5)(x_2x_3 + x_2x_4 + x_3x_4)\end{aligned}$$

is decomposable in  $R_{\mathbb{F}}[x_1, \dots, x_5]$ . However, if we cut out a vertex of  $P$ , then the resulting polytope



is not a product, and the corresponding  $\sigma_3^{K_P}(x_1, \dots, x_5, x_6) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 + x_1x_4x_6 + x_3x_4x_6 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5$  is indecomposable in  $R_{\mathbb{F}}[x_1, \dots, x_6]$ .

**Remark 2.** We note that the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R[x_1, \dots, x_m]$  is different from that of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$ . In Subsection 4.3 we shall give an example to show that the decomposability of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  in  $R(K_P)$  cannot be used to detect that of  $P$ .

#### 4. AN ALGEBRAIC CRITERION FOR $P$ TO BE $n$ -COLORABLE

**4.1.  $n$ -colorable polytopes.** Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets. We say that  $P$  is  $n$ -colorable if there is a coloring map  $c : \mathcal{F}(P) \rightarrow [n] = \{1, \dots, n\}$  such that  $c(F_i) \neq c(F_j)$  whenever  $F_i \cap F_j \neq \emptyset$ , where  $\mathcal{F}(P)$  is the set of all facets of  $P$ . An equivalence definition for  $P$  to be  $n$ -colorable is also given by saying that there is a

non-degenerate simplicial map  $K_P \rightarrow 2^{[n]}$ , where  $2^{[n]}$  is the complex determined by all faces of the simplex  $\Delta^{n-1}$  of dimension  $n - 1$ .

All  $n$ -colorable polytopes can produce an important class of quasitoric manifolds and small covers, which were studied and named as *pullbacks from the linear model* in [7]. In terms of characteristic functions, it is easy to see that  $P$  is  $n$ -colorable if and only if there is a characteristic function  $\lambda : \mathcal{F}(P) \rightarrow R_{\mathbb{F}}^n$  such that the image  $\text{Im } \lambda = \{e_1, \dots, e_n\}$ , where  $\{e_1, \dots, e_n\}$  is a basis of  $R_{\mathbb{F}}^n$ .

In terms of combinatorics, Joswig gave the following criterion in [14].

**Theorem 4.1** ([14]). *An  $n$ -dimensional simple polytope  $P$  is  $n$ -colorable if and only if every 2-face has an even number of edges.*

More recently, a criterion in terms of self-dual binary codes of  $P$  has been given in [6].

**4.2. An algebraic criterion for  $P$  to be  $n$ -colorable.** Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets.

**Theorem 4.2.** <sup>††</sup> *The simple polytope  $P$  is  $n$ -colorable if and only if  $\sigma_n^{K_P}(x_1, \dots, x_m)$  can be decomposed as  $\lambda_1 \cdots \lambda_n$  of factors of degree  $\dim \mathbb{F}$  in  $R_{\mathbb{F}}(K_P) = R_{\mathbb{F}}[x_1, \dots, x_m]/\mathcal{I}_{K_P}$ .*

*Proof.* First we note that there is a one-one correspondence between  $\mathcal{F}(P)$  and  $\{x_1, \dots, x_m\}$ . For convenience, for each facet  $F \in \mathcal{F}(P)$ , by  $x_F$  we denote the corresponding element in  $\{x_1, \dots, x_m\}$ , and conversely, for each element  $x$  in  $\{x_1, \dots, x_m\}$ , by  $F_x$  we denote the corresponding facet in  $\mathcal{F}(P)$ .

Assume that  $P$  is  $n$ -colorable. Let  $c : \mathcal{F}(P) \rightarrow [n]$  be the coloring map of  $P$ . Then there is a partition  $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$  of  $\mathcal{F}(P)$  such that  $\mathcal{F}_i = \{F \in \mathcal{F}(P) | c(F) = i\}$ . Now for each  $\mathcal{F}_i$ , set

$$\lambda_i = \sum_{F \in \mathcal{F}_i} x_F.$$

Since all facets of each  $\mathcal{F}_i$  are disjoint, this means that for each vertex  $v$  of  $P$ , there are  $n$  facets, say  $F_1, \dots, F_n$ , which come from  $\mathcal{F}_1, \dots, \mathcal{F}_n$  respectively, such that  $v = F_1 \cap \cdots \cap F_n$ . Furthermore, we see easily that

$$\sigma_n^{K_P}(x_1, \dots, x_m) = \lambda_1 \cdots \lambda_n \text{ in } R_{\mathbb{F}}(K_P)$$

as desired.

Conversely, assume that  $\sigma_n^{K_P}(x_1, \dots, x_m) = \lambda_1 \cdots \lambda_n$  in  $R(K_P)$ , where  $\deg \lambda_i = \dim \mathbb{F}$ . Since  $\sigma_n^{K_P}(x_1, \dots, x_m)$  is square-free and each monomial of  $\sigma_n^{K_P}(x_1, \dots, x_m)$  has coefficient 1 and corresponds to a vertex of  $P$ , we see that

- (a) any two different  $\lambda_i, \lambda_j$  contain no the same monomials;
- (b) each  $\lambda_i$  has the property that for any two monomials  $y, y'$  in  $\lambda_i$ ,  $yy' = 0$  in  $R_{\mathbb{F}}(K_P)$ ;
- (c) the coefficients of all monomials of each  $\lambda_i$  are 1 (this can easily be proved by performing an induction on  $n$ ).

For each  $\lambda_i$ , set

$$\mathcal{F}_i = \{F_x | x \text{ is a monomial of } \lambda_i\}.$$

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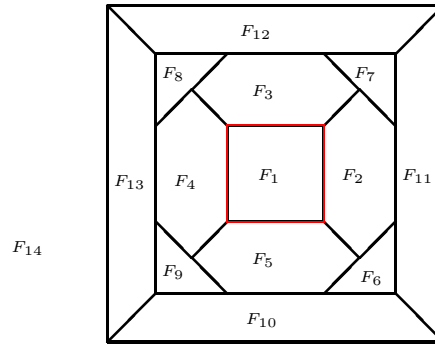
<sup>††</sup>We will see from the work of Notbohm [16, 17] stated in Subsection 4.5 that Theorem 4.2 and Theorem 4.4 below give a weaker criterion for  $P$  to be  $\ell$ -colorable than Notbohm's one, where  $\ell \geq n$ .

Then  $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$  gives a partition of  $\mathcal{F}(P)$  such that all facets of each  $\mathcal{F}_i$  are disjoint. This determines a coloring map  $c : \mathcal{F}(P) \rightarrow [n]$  defined by  $c(F) = i$  if  $F \in \mathcal{F}_i$ . Thus,  $P$  is  $n$ -colorable.  $\square$

**Corollary 4.3.** *An  $n$ -dimensional simple polytope  $P$  is  $n$ -colorable if and only if the  $n$ -th equivariant Stiefel–Whitney class  $w_n^{\mathbb{T}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{R}})$  or Chern class  $c_n^{\mathbb{T}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{C}})$  is a product  $\lambda_1 \cdots \lambda_n$  in  $H_{\mathbb{T}^m}^*(\mathcal{Z}_P^{\mathbb{R}}; R_{\mathbb{F}})$ , where  $\lambda_i \in H_{\mathbb{T}^m}^{\dim \mathbb{F}}(\mathcal{Z}_P^{\mathbb{R}}; R_{\mathbb{F}})$ .*

**Remark 3.** In Theorem 4.2,  $R_{\mathbb{F}}$  can be replaced by any commutative ring with unit.

**4.3. An example of 3-colorable polytopes.** The following polytope  $P$  with 14 facets can be colored by the coloring map  $c : \mathcal{F}(P) \rightarrow [3] = \{1, 2, 3\}$  satisfying that  $c^{-1}(1) = \{F_1, F_6, F_7, F_8, F_9, F_{14}\}$ ,  $c^{-1}(2) = \{F_2, F_4, F_{10}, F_{12}\}$ , and  $c^{-1}(3) = \{F_3, F_5, F_{11}, F_{13}\}$ . Thus this polytope  $P$  is 3-colorable, but obviously it is not a product, so  $\sigma_3^{K_P}(x_1, \dots, x_{14})$  is indecomposable in the polynomial ring  $R_{\mathbb{F}}[x_1, \dots, x_{14}]$ .



However, in the Stanley–Reisner face ring  $R_{\mathbb{F}}(K_P) = R_{\mathbb{F}}[x_1, \dots, x_{14}]/\mathcal{I}_{K_P}$ ,

$$\begin{aligned} & \sigma_3^{K_P}(x_1, \dots, x_{14}) \\ &= (x_1 + x_6 + x_7 + x_8 + x_9 + x_{14})(x_2 + x_4 + x_{10} + x_{12})(x_3 + x_5 + x_{11} + x_{13}) \end{aligned}$$

which is decomposable.

**4.4. More general case– $\ell$ -colorable polytopes.** Actually we can carry out our work on more general case. Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets, and let  $\ell \geq n$  be an integer. We say that  $P$  is  $\ell$ -colorable if there is a coloring map  $c : \mathcal{F}(P) \rightarrow [\ell] = \{1, \dots, \ell\}$  such that  $c(F_i) \neq c(F_j)$  whenever  $F_i \cap F_j \neq \emptyset$ , where  $\mathcal{F}(P)$  is the set of all facets of  $P$ .

Equivalently, we can also say that  $P$  is  $\ell$ -colorable if and only if there is a function  $\lambda : \mathcal{F}(P) \rightarrow R_{\mathbb{F}}^{\ell}$  such that  $\lambda(F_i) \neq \lambda(F_j)$  whenever  $F_i \cap F_j \neq \emptyset$  and the image  $\text{Im } \lambda = \{e_1, \dots, e_{\ell}\}$ , where  $\{e_1, \dots, e_{\ell}\}$  is a basis of  $R_{\mathbb{F}}^{\ell}$ . We see that  $P$  is always  $m$ -colorable. According to the construction method in [7, 1.5. The basic construction], we can use the pair  $(P, \lambda)$  to construct a closed manifold  $M_{\mathbb{F}}(P, \lambda)$  of dimension  $n + (\dim \mathbb{F} - 1)\ell$ , with an  $\mathbb{T}_{\mathbb{F}}^{\ell}$ -action. In particular, if  $\ell = m$ , then  $M_{\mathbb{F}}(P, \lambda)$  is exactly the (real) moment-angle manifold  $\mathcal{Z}_P^{\mathbb{F}}$ .

The following result is a generalization of Theorem 4.2.

**Theorem 4.4.** *Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets, and let  $\ell \geq n$  be an integer. Then  $P$  is  $\ell$ -colorable if and only if there exist  $\ell$  homogenous polynomial  $\lambda_1, \dots, \lambda_{\ell}$  such that  $\sigma_n^{K_P}(x_1, \dots, x_m) = \sigma_n(\lambda_1, \dots, \lambda_{\ell})$  in  $R(K_P) = R[x_1, \dots, x_m]/\mathcal{I}_{K_P}$ , where each  $\lambda_i$  is a linear combination of  $x_1, \dots, x_m$*

*Proof.* The proof follows closely that of Theorem 4.2 in a very similar way. We would like to leave it as an exercise to the reader.  $\square$

**4.5. The work of Notbohm in [16, 17].** For an  $n$ -colorable polytope  $P$  with  $m$  facets, Davis and Januszkiewicz in [7, 6.2] studied a pullback  $\pi_{\mathbb{F}} : M_{\mathbb{F}} \rightarrow P$  of the linear model, so the image of the characteristic function  $\lambda : \mathcal{F}(P) \rightarrow R_{\mathbb{F}}^n$  is some basis  $\{e_1, \dots, e_n\}$  of  $R_{\mathbb{F}}^n$ . Associated with the bundle  $\mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_m$  over  $E\mathbb{T}_{\mathbb{F}}^m \times_{\mathbb{T}_{\mathbb{F}}^m} \mathcal{Z}_P^{\mathbb{F}}$ , they used the sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  to construct  $n$  line bundles  $E_i = \mathbb{L}_1^{\otimes \lambda_{i1}} \otimes \dots \otimes \mathbb{L}_m^{\otimes \lambda_{im}}$ ,  $i = 1, \dots, n$ , with the first characteristic class of each  $E_i$  just being  $\lambda_i$  such that

$$E_1 \oplus \dots \oplus E_n \oplus \mathbb{F}^{m-n} \cong \mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_m$$

where each  $\lambda_i = \lambda_{i1}x_1 + \dots + \lambda_{im}x_m \in R_{\mathbb{F}}(K_P) = R_{\mathbb{F}}[x_1, \dots, x_m]/\mathcal{I}_{K_P}$  and  $\mathbb{F}^{m-n}$  denotes the trivial vector bundle of dimension  $m - n$ . In [16, 17], Notbohm generalized this to the study of the vector bundle over Davis–Januszkiewicz space  $DJ(K)$  for an  $(n - 1)$ -dimensional simplicial complex  $K$ . Notbohm constructed a particular  $n$ -dimensional complex vector bundle over the associated Davis–Januszkiewicz space whose Chern classes are given by the elementary symmetric polynomials in the generators of the Stanley–Reisner algebra. Furthermore, Notbohm showed that the isomorphism type of this complex vector bundle as well as of its realification are completely determined by its characteristic classes. As an application, Notbohm proved that coloring properties of the simplicial complex are completely determined by splitting properties of this bundle.

Together with Theorems 1.1-1.2 and Corollary 1.8 in [17], Notbohm gave the following theorem

**Theorem 4.5 (Notbohm).** *Let  $P$  be a simple polytope of dimension  $n$  with  $m$  facets, and let  $\ell \geq n$  be an integer. The following statements are equivalent.*

- (1)  $P$  is  $\ell$ -colorable.
- (2) There are  $\ell$  line bundles  $E_i$  ( $i = 1, \dots, \ell$ ) such that

$$E_1 \oplus \dots \oplus E_{\ell} \oplus \mathbb{F}^{m-\ell} \cong \mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_m.$$

- (3)  $c_{\mathbb{F}}(E_1 \oplus \dots \oplus E_{\ell}) = c_{\mathbb{F}}(\mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_m)$  where  $c_{\mathbb{F}}$  denotes the total Chern class if  $\mathbb{F} = \mathbb{C}$  and the total Stiefel–Whitney class if  $\mathbb{F} = \mathbb{R}$ .

**Remark 4.** In Theorem 4.5(3), write  $c_{\mathbb{F}}(E_i) = 1 + \lambda_i$  where  $\lambda_i \in R(K_P) = R[x_1, \dots, x_m]/\mathcal{I}_{K_P}$  with  $\deg \lambda_i = \dim \mathbb{F}$ . Then we have that

$$\prod_{i=1}^{\ell} (1 + \lambda_i) = c_{\mathbb{F}}(E_1 \oplus \dots \oplus E_{\ell}) = c_{\mathbb{F}}(\mathbb{L}_1 \oplus \dots \oplus \mathbb{L}_m) = \prod_{j=1}^m (1 + x_j)$$

so for  $i \leq n$ ,

$$\sigma_i^{K_P}(x_1, \dots, x_m) = \sigma_i(\lambda_1, \dots, \lambda_{\ell}).$$

This means that in Theorem 4.5, all  $\sigma_i^{K_P}(x_1, \dots, x_m)$ ,  $i \leq n$  would be involved for  $P$  to be  $\ell$ -colorable. However, Theorems 4.2–4.4 tell us that we only need to consider the  $n$ -th elementary symmetric polynomial  $\sigma_n^{K_P}(x_1, \dots, x_m)$ .

## 5. STANLEY–REISNER EXTERIOR FACE RING AND BUCHSTABER INVARIANT

**5.1. Stanley–Reisner exterior face ring.** Let  $K$  be a simplicial complex on vertex set  $[m] = \{1, \dots, m\}$ . Consider the exterior algebra  $\wedge_{\mathbb{Z}}[x_1, \dots, x_m]$  over  $\mathbb{Z}$ . In a similar way to Stanley–Reisner face ring, based upon the combinatorial structure of  $K$ , we may define the quotient ring of the exterior algebra  $\wedge_{\mathbb{Z}}[x_1, \dots, x_m]$

$$\mathcal{E}_{\mathbb{C}}(K) = \wedge_{\mathbb{Z}}[x_1, \dots, x_m] / \mathcal{I}_K$$

where  $\mathcal{I}_K$  is the ideal generated by those square-free monomials  $x_{i_1} \wedge \dots \wedge x_{i_r}$  for which  $\{i_1, \dots, i_r\} \notin K$ . We call  $\mathcal{E}_{\mathbb{C}}(K)$  the *Stanley–Reisner exterior face ring of  $K$* .

Clearly, each polynomial in  $\mathcal{E}_{\mathbb{C}}(K)$  is square-free. We can define a differential  $\partial$  on  $\mathcal{E}_{\mathbb{C}}(K)$  in a usual way. Then we have

**Lemma 5.1.**  *$(\mathcal{E}_{\mathbb{C}}(K), \partial)$  is isomorphic to the chain complex  $\mathcal{C}(K)$  of  $K$ .*

*Proof.* Given an orientation of  $K$ , each oriented simplex  $a = \langle i_1, \dots, i_r \rangle$  in  $K$  determines a monomial  $\mathbf{x}_a = x_{i_1} \wedge \dots \wedge x_{i_r}$  in  $\mathcal{E}_{\mathbb{C}}(K)$  such that the ordering of the product  $x_{i_1} \wedge \dots \wedge x_{i_r}$  agrees with the orientation of  $a$ . Then the required isomorphism is given by mapping  $a$  to  $\mathbf{x}_a$ .  $\square$

In the proof of Lemma 5.1, we call  $\mathbf{x}_a$  an *oriented monomial* corresponding to the oriented simplex  $a$ .

Set  $\mathcal{E}_{\mathbb{R}}(K) = \mathcal{E}_{\mathbb{C}}(K) \otimes \mathbb{Z}_2$ . This is the Stanley–Reisner exterior face ring over  $\mathbb{Z}_2$ , which is commutative. We will take  $\deg x_i = \dim \mathbb{F}$  for the generators  $x_i$  of  $\mathcal{E}_{\mathbb{F}}(K)$  in the following discussion.

Compare with Stanley–Reisner face ring  $R_{\mathbb{F}}(K)$ , if  $\mathbb{F} = \mathbb{R}$ , then it is easy to see that there is a natural module embedding  $\mathcal{E}_{\mathbb{R}}(K) \hookrightarrow R_{\mathbb{R}}(K)$ , so  $\mathcal{E}_{\mathbb{R}}(K)$  can be regarded as a submodule of  $R_{\mathbb{R}}(K)$ . On the other hand, there is a surjective projection  $R_{\mathbb{R}}(K) \rightarrow \mathcal{E}_{\mathbb{R}}(K)$ . This projection is a ring homomorphism. Thus  $\mathcal{E}_{\mathbb{R}}(K)$  is isomorphic to the quotient  $R_{\mathbb{R}}(K) / \langle x_i^2 \mid i = 1, \dots, m \rangle$  as rings.

If  $\mathbb{F} = \mathbb{C}$ , this case depends upon a choice of orientations. Given an orientation of  $K$ , consider all oriented monomials  $x_{i_1} \wedge \dots \wedge x_{i_r}$  corresponding to all oriented simplices in  $K$ . These oriented monomials form a basis of  $\mathcal{E}_{\mathbb{C}}(K)$  as a module. Fix this basis, then there is still a module embedding  $\mathcal{E}_{\mathbb{C}}(K) \hookrightarrow R_{\mathbb{C}}(K)$  defined by mapping  $x_{i_1} \wedge \dots \wedge x_{i_r}$  to  $x_{i_1} \cdots x_{i_r}$ . Of course, we can also define a projection  $R_{\mathbb{C}}(K) \rightarrow \mathcal{E}_{\mathbb{C}}(K)$  by mapping non square-free polynomials to zero and mapping square free monomials to corresponding elements of the basis. However, this projection is only a module homomorphism rather than a ring homomorphism. Thus, as graded modules,  $\mathcal{E}_{\mathbb{C}}(K)$  is isomorphic to  $R_{\mathbb{C}}(K) / \langle x_i^2 \mid i = 1, \dots, m \rangle$ . In particular,  $\mathcal{E}_{\mathbb{C}}(K)^{(2)} \cong R_{\mathbb{C}}(K)^{(2)}$ , where  $\mathcal{E}_{\mathbb{C}}(K)^{(2)}$  and  $R_{\mathbb{C}}(K)^{(2)}$  are the graded modules of degree 2 in  $\mathcal{E}_{\mathbb{C}}(K)$  and  $R_{\mathbb{C}}(K)$ , respectively.

**5.2. A criterion for Buchstaber invariant**  $s_{\mathbb{C}}(P) = m - n$ . Let  $P$  be an  $n$ -dimensional simple polytope with  $m$  facets  $F_1, \dots, F_m$ . In the Stanley–Reisner exterior face ring  $\mathcal{E}_{\mathbb{C}}(K_P) = \wedge_{\mathbb{Z}}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$ , we can still define elementary symmetric polynomials  $\sigma_i^{K_P}(x_1, \dots, x_m)$ , but here our definition of  $\sigma_i^{K_P}(x_1, \dots, x_m)$  depends upon the choices of the orientations of  $K_P$ .

Given an orientation  $\mathfrak{o}$  of  $K_P$ , define the  $i$ -th elementary symmetric polynomial as

$$\sigma_i^{K_P, \mathfrak{o}}(x_1, \dots, x_m) = \sum_{\substack{a \in K_P \\ \dim a = i-1}} \mathbf{x}_a$$

where  $\mathbf{x}_a$  is the oriented monomial of degree  $2i$  corresponding to  $a$ , as defined as before.

We shall show the following necessary and sufficient condition for  $s_{\mathbb{C}}(P) = m - n$  in terms of the decomposability of  $\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m)$ .

**Theorem 5.2.** *The Buchstaber invariant  $s_{\mathbb{C}}(P) = m - n$  if and only if there is an orientation  $\mathfrak{o}$  of  $K_P$  such that  $\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m)$  can be decomposed as the product  $\lambda_1 \wedge \dots \wedge \lambda_n$  of  $n$  factors of degree 2 in  $\mathcal{E}_{\mathbb{C}}(K_P)$ .*

*Proof.* We firstly choose the orientation  $\mathfrak{o}'$  of  $K_P$  which agrees with the ordering of its vertex set  $[m]$  (corresponding to the facet set  $\{F_1, \dots, F_m\}$  of  $P$ ).

Suppose that  $s_{\mathbb{C}}(P) = m - n$  and  $\lambda$  is a characteristic function on  $P$ . As mentioned in Subsection 2.4, it determines an  $(n \times m)$ -matrix  $\Lambda = (\lambda_{ij})$  and a sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  in  $R_{\mathbb{C}}(K_P)^{(2)}$ . For any oriented  $(n-1)$ -simplex  $a = \langle i_1, \dots, i_n \rangle \in K_P$  with  $i_1 < \dots < i_n$ , denote by  $\Lambda_a = (\lambda(F_{i_1}), \dots, \lambda(F_{i_n}))$  the corresponding  $(n \times n)$ -matrix. Then  $\det \Lambda_a = \pm 1$ . Now regarding  $\lambda_1, \dots, \lambda_n$  as the elements in  $\mathcal{E}_{\mathbb{C}}(K_P)^{(2)}$  since  $R_{\mathbb{C}}(K_P)^{(2)} \cong \mathcal{E}_{\mathbb{C}}(K_P)^{(2)}$ , let us consider the product  $\lambda_1 \wedge \dots \wedge \lambda_n$  in  $\mathcal{E}_{\mathbb{C}}(K_P)$ . Since this product  $\lambda_1 \wedge \dots \wedge \lambda_n$  is of degree  $2n$ , it is actually the linear combination of  $\mathbf{x}_a$  with integer coefficients, where  $a$  runs over all oriented  $(n-1)$ -simplices of  $K_P$  with the given orientation  $\mathfrak{o}'$ . We only need to figure out the coefficient for every  $\mathbf{x}_a$ .

Without the loss of generality, suppose that  $a_0 = \langle 1, \dots, n \rangle \in K_P$ . Then  $\lambda_i = \lambda_{i1}x_1 + \dots + \lambda_{in}x_n + \dots + \lambda_{im}x_m$ .

$$(5.1) \quad \lambda_1 \wedge \dots \wedge \lambda_n = \sum_{\substack{(1, \dots, n) \\ (i_1, \dots, i_n) \in S_n}} (\lambda_{1i_1}x_{i_1}) \wedge \dots \wedge (\lambda_{ni_n}x_{i_n}) + f_1$$

where  $S_n$  is the symmetric group of rank  $n$  and  $f_1$  is the part of  $\lambda_1 \wedge \dots \wedge \lambda_n$  which doesn't contain the monomial  $\mathbf{x}_{a_0}$ . Denote by  $f_2$  the 1st part of the right side of 5.1. Then

$$\begin{aligned} f_2 &= \sum \lambda_{1i_1} \dots \lambda_{ni_n} \cdot x_{i_1} \wedge \dots \wedge x_{i_n} \\ &= \left( \sum \text{sign} \begin{pmatrix} 1, \dots, n \\ i_1, \dots, i_n \end{pmatrix} \lambda_{1i_1} \dots \lambda_{ni_n} \right) x_1 \wedge \dots \wedge x_n \\ &= \det \Lambda_{a_0} \cdot \mathbf{x}_{a_0}. \end{aligned}$$

Similarly, we can know that the coefficient of other  $\mathbf{x}_a$  also equals to  $\det \Lambda_a$ , which is just  $\pm 1$ . So we only need to choose a proper orientation  $\mathfrak{o}$  of  $K_P$  such that all coefficients of  $\mathbf{x}_a$  corresponding to  $a$  under the chosen orientation  $\mathfrak{o}$  equal to 1. In fact, if  $\det \Lambda_a = -1$ , then we will choose the inverse orientation of  $a$ . Furthermore, under the chosen orientation  $\mathfrak{o}$ , we have that  $\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m) = \lambda_1 \wedge \dots \wedge \lambda_n$ .

Next, let us prove the converse. Assume that there is a proper orientation  $\mathfrak{o}$  of  $K_P$  such that  $\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m) = \lambda_1 \wedge \dots \wedge \lambda_n$ . Let  $\lambda_i = \lambda_{i1}x_1 + \dots + \lambda_{im}x_m$ . From the above arguments, we know that under the orientation  $\mathfrak{o}'$ ,

$$\lambda_1 \wedge \dots \wedge \lambda_n = \sum_{14} \det \Lambda_{a \mathfrak{o}'} \mathbf{x}_{a \mathfrak{o}'}$$

where  $\Lambda_{a^{o'}} = \begin{pmatrix} \lambda_{1i_1} & \cdots & \lambda_{1i_n} \\ \vdots & & \vdots \\ \lambda_{ni_1} & \cdots & \lambda_{ni_n} \end{pmatrix}$  for  $a^{o'} = \langle i_1, \dots, i_n \rangle^{o'} \in K_P$  with  $i_1 < \cdots < i_n$ . On the other hand, under the orientation  $\mathfrak{o}$ , we have that

$$\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m) = \lambda_1 \wedge \cdots \wedge \lambda_n = \sum \mathbf{x}_{a^{\mathfrak{o}}}.$$

Two different expressions of the product  $\lambda_1 \wedge \cdots \wedge \lambda_n$  under the two orientations  $\mathfrak{o}'$  and  $\mathfrak{o}$  give

$$\mathbf{x}_{a^{\mathfrak{o}}} = \det \Lambda_{a^{o'}} \mathbf{x}_{a^{o'}}$$

where two oriented  $(n-1)$ -dimensional simplices  $a^{\mathfrak{o}}$  and  $a^{o'}$  have the same underlying set. This means that  $\det \Lambda_{a^{o'}} = \pm 1$  for every  $(n-1)$ -dimensional oriented simplex  $a^{o'}$  under the orientation  $\mathfrak{o}'$  since  $\mathbf{x}_{a^{\mathfrak{o}}} = \pm \mathbf{x}_{a^{o'}}$  in  $\mathcal{E}_{\mathbb{C}}(K_P)$ . Then  $\Lambda = (\lambda_{ij})$  can just determine a characteristic function and therefore,  $s_{\mathbb{C}}(P) = m - n$ .  $\square$

**Remark 5.** Let  $V_P$  denote the vertex set of  $P$ . Then we know that there are  $|V_P|$  simplices of dimension  $n-1$  in  $K_P$ . Thus the number of possible orientations of all simplices of dimension  $n-1$  in  $K_P$  is  $2^{|V_P|}$ . It should be pointed out that if  $s_{\mathbb{C}}(K_P) = m - n$ , as shown in the proof of Theorem 5.2, we see that for any orientation  $\mathfrak{o}$  of  $K_P$ ,  $\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m)$  may not be decomposed as the product  $\lambda_1 \wedge \cdots \wedge \lambda_n$  of  $n$  factors of degree 2 in  $\mathcal{E}_{\mathbb{C}}(K_P)$ . Actually, this can be related with the geometry structure of the quasitoric manifold corresponding to  $\lambda = (\lambda_1, \dots, \lambda_n)$  with the given omniorientation (see Proposition 5.4 below).

**Corollary 5.3.** A sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  in  $R_{\mathbb{C}}(K_P)^{(2)}$  is a characteristic function on  $P^n$  if and only if there is an orientation  $\mathfrak{o}$  of  $K_P$  such that  $\sigma_n^{K_P, \mathfrak{o}}(x_1, \dots, x_m) = \lambda_1 \wedge \cdots \wedge \lambda_n$  in  $\mathcal{E}_{\mathbb{C}}(K_P)$ .

**Example 2.** Let  $P$  be a square with 4 facets  $F_1, F_2, F_3, F_4$  satisfying that  $F_1 \cap F_3 = \emptyset$  and  $F_2 \cap F_4 = \emptyset$ . Then  $K_P$  is a 1-dimensional simplicial complex on  $[4] = \{1, 2, 3, 4\}$ . Given an orientation  $\mathfrak{o}$  of  $K_P$  such that 4 oriented 1-dimensional simplices are  $\langle 1, 2 \rangle, \langle 1, 4 \rangle, \langle 3, 2 \rangle, \langle 4, 3 \rangle$ , respectively, then

$$\sigma_2^{K_P, \mathfrak{o}}(x_1, x_2, x_3, x_4) = x_1 \wedge x_2 + x_1 \wedge x_4 + x_3 \wedge x_2 + x_4 \wedge x_3.$$

Choose  $\lambda_1 = x_1 + x_2 + 2x_3 + 3x_4$  and  $\lambda_2 = 2x_1 + 3x_2 + 5x_3 + 7x_4$ . Then we have that

$$\begin{aligned} \lambda_1 \wedge \lambda_2 &= (x_1 + x_2 + 2x_3 + 3x_4) \wedge (2x_1 + 3x_2 + 5x_3 + 7x_4) \\ &= 3x_1 \wedge x_2 + 7x_1 \wedge x_4 + 2x_2 \wedge x_1 + 5x_2 \wedge x_3 + 6x_3 \wedge x_2 + 14x_3 \wedge x_4 \\ &\quad + 15x_4 \wedge x_3 + 6x_4 \wedge x_1 \\ &= 3x_1 \wedge x_2 + 7x_1 \wedge x_4 - 2x_1 \wedge x_2 + 5x_2 \wedge x_3 - 6x_2 \wedge x_3 + 14x_3 \wedge x_4 \\ &\quad - 15x_3 \wedge x_4 - 6x_1 \wedge x_4 \\ &= x_1 \wedge x_2 + x_1 \wedge x_4 + x_3 \wedge x_2 + x_4 \wedge x_3 \end{aligned}$$

so  $\sigma_2^{K_P, \mathfrak{o}}(x_1, x_2, x_3, x_4) = \lambda_1 \wedge \lambda_2$ . Thus  $s_{\mathbb{C}}(P) = 4 - 2 = 2$  and  $\lambda = (\lambda_1, \lambda_2)$  is a characteristic function on  $P$ .

**5.3. Almost complex structures on quasitoric manifolds.** As a manifold with corners,  $P$  has two possible orientations, each of which determines an orientation of  $K_P$ . If we take such an orientation  $\mathfrak{o}_P$  of  $P$ , then the other one will be the inverse orientation  $-\mathfrak{o}_P$  of  $\mathfrak{o}_P$ . These two orientations just determine two possible orientations of the underlying space  $|K_P|$  as a sphere, denoted by  $\pm \mathfrak{o}_{|K_P|}$ .

Now take the orientation  $\mathfrak{o}_{|K_P|}$  of  $K_P$  and assume that  $s_{\mathbb{C}}(P) = m - n$ . Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a characteristic function on  $P$  such that  $\lambda_i = \lambda_{i1}x_1 + \dots + \lambda_{im}x_m$ , and let  $M_{\mathbb{C}}(P, \lambda)$  be the corresponding quasitoric manifold over  $P$ . For each oriented  $(n - 1)$ -simplex  $a = \langle i_1, \dots, i_n \rangle$  of  $K_P$ , by  $\Lambda_a$  we denote the square matrix formed by  $n$  columns  $\lambda(F_{i_1}), \dots, \lambda(F_{i_n})$  in the order given by the orientation of  $a$ , where  $\lambda(F_{i_j}) = (\lambda_{1i_j}, \dots, \lambda_{ni_j})^T$ . Note that each oriented simplex  $a$  of dimension  $n - 1$  in  $K_P$  corresponds to a fixed point of  $M_{\mathbb{C}}(P, \lambda)$ , denoted by  $v_a$ . Then we have that

$$\lambda_1 \wedge \dots \wedge \lambda_n = \sum_{\substack{a \in K_P \\ \dim a = n-1}} \det \Lambda_a \mathbf{x}_a.$$

Without the loss of generality, assume that  $\langle 1, \dots, n \rangle$  is an oriented  $(n - 1)$ -simplex of  $K_P$  and  $\lambda_{\langle 1, \dots, n \rangle}$  is the identity  $(n \times n)$ -matrix. By [5, Definition 5.3],  $(P, \lambda)$  is a combinatorial quasitoric pair, and it determines an omniorientation of  $M_{\mathbb{C}}(P, \lambda)$ . However, in general,  $\lambda_1 \wedge \dots \wedge \lambda_n = \sum_{\substack{a \in K_P \\ \dim a = n-1}} \det \Lambda_a \mathbf{x}_a$  may not be equal to  $\sigma_n^{K_P, \mathfrak{o}_{|K_P|}}(x_1, \dots, x_m)$ . Indeed, we easily see that the coefficient  $\det \Lambda_a$  is exactly the sign  $\sigma(v_a)$  at the fixed point  $v_a$  (see [4, Definition B.6.2] for the definition of  $\sigma(v_a)$ ), which may be equal to 1 or  $-1$ . [4, Theorem 7.3.24] tells us that  $M_{\mathbb{C}}(P, \lambda)$  with the given omniorientation admits a  $\mathbb{T}_{\mathbb{C}}^n$ -invariant almost complex structure if and only if  $\sigma(v) = 1$  for all fixed points of  $M_{\mathbb{C}}(P, \lambda)$ . Thus by Theorem 5.2 we have

**Proposition 5.4.**  $\sigma_n^{K_P, \mathfrak{o}_{|K_P|}}(x_1, \dots, x_m)$  can be decomposed as  $\lambda_1 \wedge \dots \wedge \lambda_n$  in  $\mathcal{E}_{\mathbb{C}}(K_P)$  if and only if  $s_{\mathbb{C}}(P) = m - n$  and the corresponding quasitoric manifold  $M_{\mathbb{C}}(P, \lambda)$  admits a  $\mathbb{T}_{\mathbb{C}}^n$ -invariant almost complex structure.

Fix the orientation  $\mathfrak{o}_{|K_P|}$  of  $K_P$ , we easily see that  $\sigma_n^{K_P, \mathfrak{o}_{|K_P|}}(x_1, \dots, x_m)$  is a cycle in the chain complex  $(\mathcal{E}_{\mathbb{C}}(K_P), \partial)$ , i.e.,  $\partial \sigma_n^{K_P, \mathfrak{o}_{|K_P|}}(x_1, \dots, x_m) = 0$ . An easy argument shows that if  $\lambda_1 \wedge \dots \wedge \lambda_n$  is a cycle in  $(\mathcal{E}_{\mathbb{C}}(K_P), \partial)$ , then  $\sigma_n^{K_P, \mathfrak{o}_{|K_P|}}(x_1, \dots, x_m) = \pm \lambda_1 \wedge \dots \wedge \lambda_n$  since  $(\mathcal{E}_{\mathbb{C}}(K_P), \partial) \cong \mathcal{C}(K_P)$  as chain complexes by Lemma 5.1 and  $K_P$  is a simplicial sphere. Therefore, it follows that

**Corollary 5.5.** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a characteristic function on  $P$  and let  $M_{\mathbb{C}}(P, \lambda)$  be the corresponding quasitoric manifold over  $P$ . Then  $M_{\mathbb{C}}(P, \lambda)$  admits a  $\mathbb{T}_{\mathbb{C}}^n$ -invariant almost complex structure if and only if  $\lambda_1 \wedge \dots \wedge \lambda_n$  is a cycle in  $\mathcal{E}_{\mathbb{C}}(K_P)$ .

**Example 3.** In Example 2, we give an orientation of  $P$  such that all 1-dimensional oriented simplices of  $K_P$  are  $\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle, \langle 4, 1 \rangle$ , respectively. We still choose  $\lambda = (\lambda_1, \lambda_2)$  such that  $\lambda_1 = x_1 + x_2 + 2x_3 + 3x_4$  and  $\lambda_2 = 2x_1 + 3x_2 + 5x_3 + 7x_4$ , as stated in Example 2. Then we know that

$$\lambda_1 \wedge \lambda_2 = x_1 \wedge x_2 - x_2 \wedge x_3 - x_3 \wedge x_4 - x_4 \wedge x_1,$$

but obviously  $\lambda_1 \wedge \lambda_2$  is not a cycle in  $\mathcal{E}_{\mathbb{C}}(K_P)$ . Thus the corresponding 4-dimensional quasitoric manifold  $M_{\mathbb{C}}(P, \lambda)$  does not admit a  $\mathbb{T}_{\mathbb{C}}^4$ -invariant almost complex structure.

However, if we choose  $\lambda' = (\lambda'_1, \lambda'_2)$  such that  $\lambda'_1 = x_1 - x_3$  and  $\lambda'_2 = x_2 - x_4$ , then

$$\lambda'_1 \wedge \lambda'_2 = x_1 \wedge x_2 + x_2 \wedge x_3 + x_3 \wedge x_4 + x_4 \wedge x_1$$

so  $\lambda'_1 \wedge \lambda'_2$  is a cycle in  $\mathcal{E}_{\mathbb{C}}(K_P)$ . Thus the corresponding 4-dimensional quasitoric manifold  $M_{\mathbb{C}}(P, \lambda')$  admits a  $\mathbb{T}_{\mathbb{C}}^4$ -invariant almost complex structure.



**5.4. A criterion for Buchstaber invariant**  $s_{\mathbb{R}}(P) = m - n$ . In this case, we know from Subsection 5.1 and Theorem 2.1 that there is a natural surjective projection

$$H_{T_{\mathbb{R}}^m}^*(\mathcal{Z}_P^{\mathbb{R}}; \mathbb{Z}_2) = R_{\mathbb{R}}(K_P) \longrightarrow \mathcal{E}_{\mathbb{R}}(K_P) = H_{T_{\mathbb{R}}^m}^*(\mathcal{Z}_P^{\mathbb{R}}; \mathbb{Z}_2) / \langle x_i^2 | i = 1, \dots, m \rangle$$

where  $\mathcal{E}_{\mathbb{R}}(K_P) = \wedge_{\mathbb{Z}_2}[x_1, \dots, x_m] / \mathcal{I}_{K_P}$  with  $\deg x_i = 1$ .

There is a real analogue of Theorem 5.2. Since we do not need to consider the orientation in the real case, its proof is much easier and omitted.

**Theorem 5.6.**  $s_{\mathbb{R}}(P) = m - n$  if and only if  $w_n^{\mathbb{T}_{\mathbb{R}}^m}(\mathcal{T}\mathcal{Z}_P^{\mathbb{R}}) = \sigma_n^{K_P}(x_1, \dots, x_m)$  can be decomposed as the product  $\lambda_1 \cdots \lambda_n$  of  $n$  factors of degree 1 in  $\mathcal{E}_{\mathbb{R}}(K_P) = H_{T_{\mathbb{R}}^m}^*(\mathcal{Z}_P^{\mathbb{R}}; \mathbb{Z}_2) / \langle x_i^2 | i = 1, \dots, m \rangle$ .

**Corollary 5.7.** The set

$$\{\lambda = (\lambda_1, \dots, \lambda_n) \text{ in } H_{T_{\mathbb{R}}^m}^1(\mathcal{Z}_P^{\mathbb{R}}; \mathbb{Z}_2) \mid \sigma_n^{K_P}(x_1, \dots, x_m) = \lambda_1 \cdots \lambda_n \text{ in } \mathcal{E}_{\mathbb{R}}(K_P)\}$$

gives all characteristic functions on  $P^n$ .

**5.5. An application to cyclic polytopes.** The moment curve in  $\mathbb{R}^n$  is given by  $v(t) = (t, t^2, \dots, t^n)$ . For any  $m > n$ , define the cyclic polytope  $C^n(m)$  as the convex hull of  $m$  distinct points  $v(t_i)$ ,  $t_1 < t_2 < \dots < t_m$  on the moment curve in  $\mathbb{R}^n$ .  $C^n(m)$  is a simplicial polytope, i.e. its boundary is a simplicial  $(n - 1)$ -sphere, still denoted by  $C^n(m)$ .

In the case of  $n \geq 4$ , it has been already shown in [7] that when  $m \geq 2^n$ ,  $s_{\mathbb{F}}(C^n(m)) < m - n$ , since  $C^n(m)$  is 2-neighbourly. But actually the precise lower bound for  $m$  is much smaller.

First let us look at the case  $n = 4$ .

**Lemma 5.8.** When  $m \geq 8$ ,  $s_{\mathbb{R}}(C^4(m)) < m - 4$ . In particular, when  $m = 8$ ,  $s_{\mathbb{R}}(C^4(m)) = m - 5$ , and when  $m = 7$ ,  $s_{\mathbb{R}}(C^4(m)) = m - 4$ .

*Proof.* For a convenience, write  $v_i = v(i)$  where  $i = 1, \dots, m$ . According to [18], if  $a = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\} \in C^4(m)$  where  $i_1 < \dots < i_4$ , then there are exactly 2 possibilities:

- (1)  $i_1 = 1, i_4 = m, i_3 - i_2 = 1$ .
- (2)  $i_2 - i_1 = i_4 - i_3 = 1$ .

Suppose there exists a characteristic function  $\lambda : \{v_i\}_{1 \leq i \leq m} \rightarrow \mathbb{Z}_2^4$ . Then by Theorem 5.6,

$$\sigma_4^{C^4(m)}(x_1, \dots, x_m) = \lambda_1 \cdots \lambda_4,$$

where  $\lambda_i = \lambda_{i1}x_1 + \dots + \lambda_{im}x_m$  are elements of degree 1 in  $\mathcal{E}_{\mathbb{R}}(C^4(m))$ .

We claim:

- (1)  $(\lambda_{1i}, \dots, \lambda_{4i}) \neq (\lambda_{1j}, \dots, \lambda_{4j})$ , if  $i \neq j$ .
- (2)  $(\lambda_{1i}, \dots, \lambda_{4i}) + (\lambda_{1j}, \dots, \lambda_{4j}) \neq (\lambda_{1k}, \dots, \lambda_{4k})$ , if  $i - j \equiv 1$  or  $m - 1 \pmod{m}$ , and  $k \neq i, j$ .
- (3)  $(\lambda_{1i}, \dots, \lambda_{4i}) + (\lambda_{1j}, \dots, \lambda_{4j}) \neq (\lambda_{1k}, \dots, \lambda_{4k}) + (\lambda_{1l}, \dots, \lambda_{4l})$ , if  $i - j \equiv k - l \equiv 1$  or  $m - 1 \pmod{m}$ , and  $i, j, k, l$  are distinct.

If Claim (1) is not true, without loss of generality, assume  $(\lambda_{11}, \dots, \lambda_{41}) = (\lambda_{12}, \dots, \lambda_{42})$ , then for each  $i$ ,

$$\lambda_i = \lambda_{i1}(x_1 + x_2) + \lambda_{i3}x_3 + \dots + \lambda_{im}x_m.$$

Since  $(x_1 + x_2)^2 = 0$  in  $\mathcal{E}_{\mathbb{R}}(C^4(m))$ , the monomials in  $\sigma_4^{C^4(m)}(x_1, \dots, x_m)$  containing  $x_1x_2$  must vanish. This is contradictory. If Claim (2) is not true, also assume  $(\lambda_{11}, \dots, \lambda_{41}) + (\lambda_{12}, \dots, \lambda_{42}) = (\lambda_{1k}, \dots, \lambda_{4k})$ ,  $k \neq 1, 2$ , then

$$\lambda_i = \lambda_{i1}(x_1 + x_k) + \lambda_{i2}(x_2 + x_k) + \lambda_{i3}x_3 + \dots + \lambda_{im}x_m.$$

The monomials containing  $x_1x_2x_k$  must be obtained from multiplying by factor  $(x_1 + x_k)^2$  or  $(x_2 + x_k)^2$ , which vanishes. It is similar to verify Claim (3).

So by the above three claims, the two sets  $\{(\lambda_{1i}, \dots, \lambda_{4i}) : i = 1, \dots, m\}$  and

$$\{(\lambda_{1i}, \dots, \lambda_{4i}) + (\lambda_{1j}, \dots, \lambda_{4j}) : i - j \equiv 1 \text{ or } m - 1 \pmod{m}\}$$

occupy  $2m$  different non-zero elements in  $\mathbb{Z}_2^4$ , which is impossible when  $m \geq 8$ .

Actually,  $m = 8$  is the precise lower bound satisfying  $s_{\mathbb{R}}(C^4(m)) < m - 4$  for the case of  $n = 4$ . We can give the specific matrices determined by the characteristic functions of  $C^4(7)$  and  $C^4(8)$  as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

for  $C^4(7)$ , and

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

for  $C^4(8)$ , respectively. Note that the above matrix for  $C^4(7)$  also means that

$$\sigma_4^{C^4(7)}(x_1, \dots, x_7) = (x_1 + x_5 + x_6)(x_2 + x_6 + x_7)(x_3 + x_5 + x_6 + x_7)(x_4 + x_5 + x_7)$$

in  $\mathcal{E}_{\mathbb{R}}(C^4(7))$ . Thus,  $s_{\mathbb{R}}(C^4(7)) = s_{\mathbb{R}}(C^4(8)) = 3$ . □

In the general case, we have:

**Proposition 5.9.** *Let  $n \geq 4$ . When  $m \geq 2^{n-1}$ ,  $s_{\mathbb{R}}(C^n(m)) < m - n$ .*

*Proof.* With the above notation, as shown in [18], the criterion for the 3-faces of  $C^n(m)$  with  $n \geq 4$  is compatible. This means that  $a = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$  with  $i_1 < \dots < i_4$  satisfies either of the following conditions:

- (1)  $i_1 = 1, i_4 = m, i_3 - i_2 = 1$ ;
- (2)  $i_2 - i_1 = i_4 - i_3 = 1$ ,

if and only if  $a$  is a 3-face in  $C^n(m)$ . So the argument of Lemma 5.8 can be still applied to the case of higher dimensions. □

**Corollary 5.10.** *Let  $n \geq 4$ . If  $m \geq 2^{n-1}$ , then  $s_{\mathbb{C}}(C^n(m)) < m - n$ .*

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